RAN-2103000206020031

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T.Y. B.Sc. (Sem. VI) Examination March - 2025

Mathematics (Paper : MTH - 601)

Ring Theory

Time: 2 Hours] **[Total Marks: 50**

સૂચના : / Instructions

(૧) **નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.**
Fill up strictly the details of ☛ signs on your answer book

Name of the Examination:

☛ **T.Y. B.Sc. (Sem. VI)**

Name of the Subject :

☛ **Mathematics (Paper : MTH - 601) Ring Theory**

Subject Code No.: **2103000206020031**

Seat No.:

Student's Signature

(2) All questions are compulsory.

(3) Figures to the right indicate marks of the questions.

(4) Follow usual notations.

Q. 1. Answer the following as directed: (Any FIVE) **10**

1. Give an example of:
   i. a ring commutative having 2025 elements;
   ii. a commutative ring without a *unit* element.
2. In a ring R; prove that $(-a) \, . \, b = -(a.b)$; for all a, b in R.
3. If U is an ideal of a ring R with a *unit* element 1 and $1 \in U$, then prove that $U = R$.
4. Define Maximal Ideal of Ring. Which is the maximal ideal of a field F ?
5. Show that 3 does not divide 3 itself and 6 does not divide 12 in the commutative ring $3\mathbb{Z}$.
6. Find all the *units* in the commutative ring J_9; of integers modulo 9; with a *unit* element $\overline{1}$.

7. Define Prime Element in a Euclidean Ring. Mention all the prime elements in the Euclidean rings $\mathbb{Z}$ of all integers.
8. Define Relatively Prime Elements in a Euclidean Ring. Give an illustration of a Euclidean Ring; in which every pair of two non-zero elements are relatively prime elements.

Q. 2. Answer any Two of the following: **10**

a. Define Field. Answer as directed:

i. Find the *Unit* element in the commutative ring $\langle R = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\}, +_{10}, \times_{10}\rangle$.

ii. Is $\langle R = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\}, +_{10}, \times_{10}\rangle$ integral domain? Justify your answer.

b. Define Integral Domain. Prove that every field is an integral domain.

c. Give an illustration of a Boolean ring. Prove that every Boolean ring is commutative.

Q. 3. Answer any Two of the following: **10**

a. Define the Kernel of Homomorphism $\phi : R \rightarrow R'$ of a ring *R* into a ring *R*'. Let $\phi : R \rightarrow R'$ be a homomorphism of a ring *R* into a ring *R*'. Then prove that: (i) $\phi(0) = 0'$; (ii) $\phi(-a) = -\phi(a)$; for every a in *R*.

b. Mention all the ideals of the commutative ring J_7; of integers modulo 7. If *F* is a field, then prove that its only ideals are (0) and *F* itself.

c. Let $\phi : R \rightarrow R'$ be a homomorphism of a ring *R* onto a ring *R*'. If *R* is a commutative ring with a *unit* element l, then prove that *R*' is also a commutative ring with a *unit* element ϕ(l).

Q. 4. Answer any Two of the following: **10**

a. Give an illustrate of a Euclidean ring; which is not a field. Give the reason why the commutative ring:

i. J_{2025}; of integers modulo 2025; is not a Euclidean ring,

ii. J_{47}; of integers modulo 47; is a Euclidean ring.

b. Find all *Units* in the commutative ring $\langle R = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}\}, +_{20}, \times_{20}\rangle$; with the unit element $\overline{16}$. If; for a, b in an integral domain R with a *unit* element; both $a \mid b$ and $b \mid a$ hold true, then prove that $a = u\, b$; where u is *unit* in R.

c. Let R be a Euclidean ring and $a \neq 0$, $b \neq 0$ in R. If b is not *unit* in R, then prove that $d(a) < d\,(ab)$.

Q. 5. Answer any Two of the following: 10

a. Prove, in a Euclidean ring R, that:

i. If a,b and c are elements in R such that $(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

ii. Let $a \neq 0$, $b \neq 0$ in R. If b is *unit* in R. Then $d(a) = d(ab)$.

b. Prove that an element a in a Euclidean ring R is *unit* in R if and only if $d(a) = d\,(1)$.

c. Prove, in a Euclidean ring R, that:

i. If π is a prime element in R, then for any element a either (π, a) *unit* in R or $\pi \mid a$.

ii. Let a, b be elements in R. If $(a, b) = d_1$ and $(a, b) = d_2$, then d_1 and d_2 are *associates*.
